On the median number of P_3's in $G(n,p)$

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Let k, b be positive integers, $b \leq k$. Let $\xi_{b,k} \sim \text{bin}(k, b/k)$, $\eta_b \sim \text{pois}(b)$. Moreover, set $p_{b,k} = \mathsf{P}(\xi_{b,k} < b)$, $p_b = \mathsf{P}(\eta_b < b)$. The number b is the median of $\text{bin}(k, b/k)$ and $\text{pois}(b)$. By Poisson limit theorem, $p_{b,k} \to p_b$ as $k \to \infty$. The question is, how close are the probabilities $p_{b,k}, p_b$ to $1/2$?

Define a sequence y_b in the following way: $\frac{1}{2} = p_b + y_b \mathsf{P}(\eta_b = b)$. Then [3, 4, 5], for all b, $\frac{1}{3} < y_b \leq \frac{1}{2}$ and y_b decreases to $\frac{1}{3}$ as $b \to \infty$. In particular, from this result it follows that $z_{b,k}$ decreases (as the function of b) for k large enough.

For any positive integer b, define a sequence $z_{b,k}$, $k \in \mathbb{N}$, $k > 2b$, in the following way: $\frac{1}{2} = p_{b,k} + z_{b,k} \mathsf{P}(\xi_{b,k} = b)$. Then [2], for every b, $z_{b,k}$ decreases for $k \geq 2b$ and $z_{b,k} \to y_b$ as $k \to \infty$. Moreover, for all $k > 2b$, $\frac{1}{3} < z_{b,k} < \frac{1}{2}$. For all $b < k < 2b$, $\frac{1}{2} < z_{b,k} < \frac{2}{3}$. Finally, $z_{b,2b} = \frac{1}{2}$.

In contrast, we study a behavior of distributions of sums of *dependent* Bernoulli random variables "near" its medians. Let $p = cn^{-3/2}$, X_n — the number of P_3's in $G(n,p)$. From the theorem about Poisson approximations for the numbers of copies of fixed graphs [1], X_n converges to a Poisson random variable with the parameter $c^2/2$. Let

$$\frac{1}{2} = \mathsf{P}(X_n < b) + w_c^n \mathsf{P}(X_n = b).$$

We prove that, *for every positive c and every non-negative integer b*, for n large enough, the probability $\mathsf{P}(X_n = b)$ decreases as the function of n. Thus, in contrast with the behavior of $z_{b,k}$, w_c^n increases for n large enough. From our result, we immediately get the following. Let μ_c be the median of Poisson distribution with the parameter $c^2/2$. Then, for n large enough, the median of X_n equals μ_c.

References

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